

Existence and multiplicity of solutions for a nonlinear Schrödinger equation with non-local regional diffusion

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Abstract

In this article we are interested in the following non-linear Schrödinger equation with non-local regional diffusion

$$(-\Delta)_{\rho_\epsilon}^\alpha u + u = f(u) \text{ in } \mathbb{R}^n, \quad u \in H^\alpha(\mathbb{R}^n), \quad (P_\epsilon)$$

where $\epsilon > 0$, $0 < \alpha < 1$, $(-\Delta)_{\rho_\epsilon}^\alpha$ is a variational version of the regional laplacian, whose range of scope is a ball with radius $\rho_\epsilon(x) = \rho(\epsilon x) > 0$, where ρ is a continuous function. We give general conditions on ρ and f which assure the existence and multiplicity of solution for (P_ϵ) .

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1 Introduction

The aim of this article is to study the non-linear Schrödinger equation with non-local regional diffusion

$$(-\Delta)_{\rho_\epsilon}^\alpha u + u = f(u) \quad \text{in } \mathbb{R}^n, \quad u \in H^\alpha(\mathbb{R}^n), \quad (P_\epsilon)$$

where $\epsilon > 0$, $0 < \alpha < 1$, $n \geq 2$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 function. The operator $(-\Delta)_{\rho_\epsilon}^\alpha$ is a variational version of the non-local regional laplacian, with range of scope determined by function $\rho_\epsilon(x) = \rho(\epsilon x)$, where $\rho \in C(\mathbb{R}^n, (0, +\infty))$.

As pointed out in [12], when studying the singularly perturbed equation (see equation (1.3) below), the scope function ρ , that describes the size of the ball of the influential region of the non-local operator, plays a key role in deciding the concentration point of ground states of the equation. Even though, at a first sight, the minimum point of ρ seems to be the concentration point, there is a non-local effect that needs to be taken in account.

Recently, a great attention has been focused on the study of problems involving the fractional Laplacian, from a pure mathematical point of view as well as from concrete applications, since this operator naturally arises in many different contexts, such as obstacle problems, financial mathematics, phase transitions, anomalous diffusions, crystal dislocations, soft thin films, semipermeable membranes, flame propagations, conservation laws, ultra relativistic limits of quantum mechanics, quasi-geostrophic flows, minimal surfaces, materials science and water waves. The literature is too wide to attempt a reasonable list of references here, so we derive the reader to the work by Di Nezza, Patalluci and Valdinoci [3], where a more extensive bibliography and an introduction to the subject are given.

In the context of fractional quantum mechanics, non-linear fractional Schrödinger equation has been proposed by Laskin [19], [20] as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. In the last 10 years, there has been a lot of interest in the study of the fractional Schrödinger equation, see the works in [6], [7], [10], [16] and [25]. In a recent paper Felmer, Quaas and Tan [10] considered positive solutions of nonlinear fractional Schrödinger equation

$$(-\Delta)^\alpha u + u = f(x, u) \text{ in } \mathbb{R}^n. \quad (1.1)$$

They obtained the existence of a ground state by mountain pass argument and a comparison method devised by Rabinowitz in [28] for $\alpha = 1$. They analyzed regularity, decay and symmetry properties of these solutions. At this point it is worth mentioning that the uniqueness of the ground state of $(-\Delta)^\alpha u + u = u^{p+1}$ in \mathbb{R} for general $\alpha \in (0, 1)$, where $0 < p < 4\alpha/(1-2\alpha)$ for $\alpha \in (0, \frac{1}{2})$ and $0 < p < \infty$ for $\alpha \in [\frac{1}{2}, 1)$, was proved by Frank and Lenzmann in [9]. Recently, the result of [9] has been extended in any dimension when α is sufficiently close to 1 by Fall and Valdinoci in [8] and later for general $\alpha \in (0, 1)$ by Frank, Lenzmann and Silvestre in [13]. We also mention the work by Cheng [6], where the fractional Schrödinger equation

$$(-\Delta)^\alpha u + V(x)u = u^p \text{ in } \mathbb{R}^n \quad (1.2)$$

with unbounded potential V was studied. The existence of a ground state

of (1.2) is obtained by Lagrange multiplier method and the Nehari manifold method is used to obtain standing waves with prescribed frequency.

On the other hand, research has been done in recent years regarding regional fractional laplacian, where the scope of the operator is restricted to a variable region near each point. We mention the work by Guan [14] and Guan and Ma [15] where they study these operators, their relation with stochastic processes and they develop integration by parts formula, and the work by Ishii and Nakamura [17], where the authors studied the Dirichlet problem for regional fractional Laplacian modeled on the p -Laplacian.

Very recently Felmer and Torres [11, 12], considered positive solutions of nonlinear Schrödinger equation with non-local regional diffusion

$$\epsilon^{2\alpha}(-\Delta)_\rho^\alpha u + u = f(u) \quad \text{in } \mathbb{R}^n, \quad u \in H^\alpha(\mathbb{R}^n). \quad (1.3)$$

The operator $(-\Delta)_\rho^\alpha$ is a variational version of the non-local regional Laplacian, defined by

$$\int_{\mathbb{R}^n} (-\Delta)_\rho^\alpha u v dx = \int_{\mathbb{R}^n} \int_{B(0, \rho(x))} \frac{[u(x+z) - u(x)][v(x+z) - v(x)]}{|z|^{n+2\alpha}} dz dx.$$

Under suitable assumptions on the nonlinearity f and the range of scope ρ , they obtained the existence of a ground state by mountain pass argument and a comparison method devised by Rabinowitz in [28] for $\alpha = 1$. Furthermore, they analyzed symmetry properties and concentration phenomena of these solutions. These regional operators present various interesting characteristics that make them very attractive from the point of view of mathematical theory of non-local operators.

Furthermore, in a recent paper [26], Pu, Liu and Tang have considered the problem

$$(-\Delta)_\rho^\alpha u + V(x)u = f(u, x) \quad \text{in } \mathbb{R}^n, \quad u \in H^\alpha(\mathbb{R}^n), \quad (1.4)$$

by assuming that ρ and V are bounded from below and there exist $r_0 > 0$ such that for any $M > 0$,

$$\lim_{|y| \rightarrow \infty} \text{meas}(\{x \in \mathbb{R}^n : |x - y| \leq r_0, V(x) \leq M\}) = 0,$$

and the nonlinearity $f(x, u)$ satisfy suitable condition, they have proved the existence of a nonnegative ground state solution for (1.4). Moreover, we should mention that the Dirichlet boundary value problem on a bounded domain with regional diffusion were investigated by the second author in [29].

Motivated by these previous results, in this paper we intend to consider new class of functions ρ , more precisely we will consider the following classes:

Class 1: ρ is periodic

(ρ_1) $\rho \in C(\mathbb{R}^n, (0, +\infty))$ and

$$0 < \rho_0 = \inf_{x \in \mathbb{R}^n} \rho(x).$$

(ρ_2) $\rho(x + T) = \rho(x)$, $x \in \mathbb{R}^n$, $T \in \mathbb{Z}^n$.

Class 2: ρ is asymptotically periodic

(ρ_3) There is a continuous periodic function $h_\infty : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$0 < \rho_0 = \inf_{x \in \mathbb{R}^n} \rho(x) \leq \rho(x) \leq h_\infty(x) \quad \forall x \in \mathbb{R}^n.$$

(ρ_4)

$$|h_\infty(x) - \rho(x)| \rightarrow 0 \quad \text{as } |x| \rightarrow +\infty.$$

Class 3: ρ has finite global minimum points

The function ρ verifies (ρ_1) and

(ρ_5)

$$\rho_\infty = \lim_{|x| \rightarrow +\infty} \rho(x) > \rho(x), \quad \forall x \in \mathbb{R}^n.$$

(ρ_6) There are only l points $a_1, a_2, \dots, a_l \in \mathbb{R}^n$ such that

$$\rho(a_i) = \inf_{x \in \mathbb{R}^n} \rho(x), \quad \forall i \in \{1, \dots, l\}.$$

Without lost of generality, we will assume that

$$\inf_{x \in \mathbb{R}^n} \rho(x) = 1 \quad \text{and} \quad a_1 = 0.$$

Associated with the function f , we assume the following conditions:

(f_1) $f \in C^1(\mathbb{R}, \mathbb{R})$ and

$$\lim_{|t| \rightarrow \infty} \frac{f(t)}{|t|^{q-1}} = 0, \quad \lim_{|t| \rightarrow \infty} \frac{f(t)t}{|t|^2} = +\infty$$

for some $q \in (2, 2_\alpha^*)$, where $2_\alpha^* = \frac{2n}{n-2\alpha}$.

(f₂) $f(t) = o(|t|)$, as $|t| \rightarrow 0$.

(f₃) There exists $\theta \geq 1$ such that $\theta \mathcal{F}(t) \geq \mathcal{F}(\sigma t)$ for $t \in \mathbb{R}$ and $\sigma \in [0, 1]$, where

$$\mathcal{F}(t) = f(t)t - 2F(t), \text{ where } F(t) = \int_0^t f(s)ds.$$

Now we are in a position to state our main existence theorem.

Theorem 1.1. *Assume $0 < \alpha < 1$, $n \geq 2$ and $(f_1) - (f_3)$. If $\epsilon = 1$ and*

i) ρ belongs to Class 1

or

ii) ρ belongs to Class 2 and f also satisfies

(f₄) $\frac{f(t)}{|t|}$ is strictly increasing in t ,

then problem (P_ϵ) possesses a non-trivial weak solution. Moreover, if ρ belongs to Class 3 and f satisfies $(f_1), (f_2), (f_4)$ and

(f'₃) There exists $\theta > 2$ such that

$$0 < \theta F(t) \leq f(t)t \text{ where } F(t) = \int_0^t f(s)ds,$$

then there is $\epsilon_ > 0$, such that problem (P_ϵ) has at least l non-trivial weak solutions for $\epsilon \in (0, \epsilon_*)$.*

Before concluding this introduction, we would like point out that in the proof of Theorem 1.1 we adapt some ideas explored in Alves, Carrião & Miyagaki [2], Cao & Noussair [4], Cao & Zhou [5], Hsu, Lin & Hu [22], Lin [23] and Hu & Tang [24]. In the above papers the authors have studied the existence and multiplicity of solution for problems involving the Laplacian operator.

The plan of the paper is as follows: In Section 2, we review some properties of the function space which will be used. In Section 3, we prove some technical lemmas in while in Section 4 we prove the main result. Finally, in Section 5 we write a remark about the existence of ground state solution.

2 Preliminaries

The fractional Sobolev space of order α on \mathbb{R}^n is defined by

$$H^\alpha(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(z)|^2}{|x - z|^{n+2\alpha}} dz dx < \infty \right\},$$

endowed with the norm

$$\|u\|_\alpha = \left(\int_{\mathbb{R}^n} |u(x)|^2 dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(z)|^2}{|x - z|^{n+2\alpha}} dz dx \right)^{1/2}.$$

Given a function ρ as above, we define

$$\|u\|^2 = \int_{\mathbb{R}^n} \int_{B(0, \rho(x))} \frac{|u(x+z) - u(x)|^2}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} |u(x)|^2 dx. \quad (2.1)$$

and the space

$$H_\rho^\alpha(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \|u\|^2 < \infty\}.$$

We note that, if ρ satisfies (ρ_1) , there exists a constant $\tilde{C} > 0$ such that

$$\|u\|_\alpha \leq \tilde{C} \|u\|.$$

This inequality implies that $H_\rho^\alpha(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ is continuous for any $q \in [2, 2_\alpha^*]$ and $H_\rho^\alpha(\mathbb{R}^n) \hookrightarrow L_{loc}^q(\mathbb{R}^n)$ is compact for any $q \in [2, 2_\alpha^*)$ (for more details, see [12]). From the above remark, we ensure that

$$H^\alpha(\mathbb{R}^n) = H_\rho^\alpha(\mathbb{R}^n) = H_{\rho_\epsilon}^\alpha(\mathbb{R}^n) = H_{h_\infty}^\alpha(\mathbb{R}^n).$$

Moreover, the norms $\|\cdot\|_\alpha$, $\|\cdot\|$ and

$$\|u\|_\infty^2 = \int_{\mathbb{R}^n} \int_{B(0, h_\infty(x))} \frac{|u(x+z) - u(x)|^2}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} |u(x)|^2 dx.$$

are equivalents on $H^\alpha(\mathbb{R}^n)$.

We would like point out that if ρ is a \mathbb{Z}^n -periodic function and $y \in \mathbb{Z}^n$, a simple change variable gives

$$\int_{\mathbb{R}^n} \int_{B(0, \rho(x+y))} \frac{|u(x+z) - u(x)|^2}{|z|^{n+2\alpha}} dz dx = \int_{\mathbb{R}^n} \int_{B(0, \rho(x))} \frac{|u(x+z) - u(x)|^2}{|z|^{n+2\alpha}} dz dx.$$

The above equality will be used frequently in our paper.

The following lemma is a version of the concentration compactness principle proved by Felmer and Torres [12].

Lemma 2.1. *Let $n \geq 2$. Assume that $\{u_k\}$ is bounded in $H_\rho^\alpha(\mathbb{R}^n)$ with*

$$\lim_{k \rightarrow \infty} \sup_{y \in \mathbb{R}^n} \int_{B(y, R)} |u_k(x)|^2 dx = 0,$$

for some $R > 0$. Then $u_k \rightarrow 0$ in $L^q(\mathbb{R}^n)$ for $q \in (2, 2_\alpha^)$.*

Associated with (P_ϵ) we have the functional $I : H_{\rho_\epsilon}^\alpha(\mathbb{R}^n) \rightarrow \mathbb{R}$ defined by

$$I(u) = \frac{1}{2} \left(\int_{\mathbb{R}^n} \int_{B(0, \rho_\epsilon(x))} \frac{|u(x+z) - u(x)|^2}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} |u(x)|^2 dx \right) - \int_{\mathbb{R}^n} F(u(x)) dx. \quad (2.2)$$

From (f_1) , $I \in C^1(H_{\rho_\epsilon}^\alpha(\mathbb{R}^n), \mathbb{R})$ with its Fréchet derivative given by

$$\begin{aligned} I'(u)v &= \int_{\mathbb{R}^n} \int_{B(0, \rho_\epsilon)} \frac{[u(x+z) - u(x)][v(x+z) - v(x)]}{|z|^{n+2\alpha}} + \int_{\mathbb{R}^n} u(x)v(x) dx \\ &\quad - \int_{\mathbb{R}^n} f(u(x))v(x) dx, \end{aligned}$$

for $u, v \in H_{\rho_\epsilon}^\alpha(\mathbb{R}^n)$. Therefore, the critical points of I are weak solutions of (P_ϵ) .

3 Technical lemmas

In this section, we are going to prove some technical results, for that purpose we take borrow some ideas of [12] and [21]. First all, we would like point out the following properties involving the function f :

Property 3.1. *1. By condition (f_1) and (f_2) , for any $\tau > 0$ there exists a constant $C_\tau > 0$ such that*

$$|F(t)| \leq \tau |t|^2 + C_\tau |t|^q. \quad (3.1)$$

2. By (f_3) , we deduce that

$$\mathcal{F}(t) = f(t)t - 2F(t) \geq 0, \text{ for all } t \in \mathbb{R}.$$

Furthermore, if $t > 0$ then we have

$$\frac{\partial}{\partial t} \left(\frac{F(t)}{t^2} \right) = \frac{tf(t) - 2F(t)}{t^3} \geq 0. \quad (3.2)$$

By (f_2) ,

$$\lim_{t \rightarrow 0^+} \frac{F(t)}{t^2} = 0. \quad (3.3)$$

Next, from (3.2) and (3.3), we conclude that $F(t) \geq 0$ for all $t \in \mathbb{R}$.

Using the above properties we are ready to prove our technical results.

Lemma 3.1. *The functional I satisfies the mountain pass geometry.*

Proof. By (3.1),

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - \tau\|u\|_{L^2}^2 - C_\tau\|u\|_{L^q}^q \\ &\geq \left(\frac{1}{2} - \tau C_2\right)\|u\|^2 - C_\tau C_q\|u\|^q. \end{aligned}$$

Let $\tau > 0$ small enough such that $\frac{1}{2} - \tau C_2 > 0$ and $\|u\| = \zeta$. Since $q > 2$, we can take ζ small enough such that

$$\frac{1}{2} - \tau C_2 - C_\tau C_q \zeta^{q-2} > 0.$$

Therefore

$$I(u) \geq \zeta^2 \left(\frac{1}{2} - \tau C_2 - C_\tau C_q \zeta^{q-2} \right) := \beta > 0.$$

Now, by (f_1) ,

$$\lim_{|t| \rightarrow \infty} \frac{F(t)}{|t|^2} = +\infty.$$

Then, for $\varphi \in C_0^\infty(\mathbb{R}^n) \setminus \{0\}$,

$$\lim_{|t| \rightarrow \infty} \int_{\mathbb{R}^n} \frac{F(t\varphi)}{|t|^2} dx = +\infty.$$

Consequently,

$$\frac{I(t\varphi)}{|t|^2} = \frac{1}{2}\|\varphi\|^2 - \int_{\mathbb{R}^n} \frac{F(t\varphi)}{|t|^2} dx \rightarrow -\infty, \text{ as } |t| \rightarrow \infty.$$

Thereby, setting $t_0 > 0$ large enough and $e = t_0\varphi$, we have $I(e) < 0$. \square

Lemma 3.2. *Assume $(f_1) - (f_2)$, $\epsilon = 1$ and that ρ belongs to Class 1 or 2. Let $c \in \mathbb{R}$ and $\{u_k\} \subset H_\rho^\alpha(\mathbb{R}^n)$ be a sequence such that*

$$I(u_k) \rightarrow c \quad \text{and} \quad I'(u_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (3.4)$$

Then $\{u_k\}$ is bounded in $H_\rho^\alpha(\mathbb{R}^n)$.

Proof. To begin with, we recall that

$$\rho_0 \leq \rho(x) \leq \rho_*, \quad \forall x \in \mathbb{R}^n,$$

for $\rho_* = \sup_{x \in \mathbb{R}^n} \rho(x)$. Hence, the functions below

$$\|u\|_0 = \left(\int_{\mathbb{R}^n} \int_{B(0, \rho_0)} \frac{|u(x+z) - u(x)|^2}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} |u(x)|^2 dx \right)^{\frac{1}{2}}$$

and

$$\|u\|_* = \left(\int_{\mathbb{R}^n} \int_{B(0, \rho_*)} \frac{|u(x+z) - u(x)|^2}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} |u(x)|^2 dx \right)^{\frac{1}{2}}$$

are equivalent norms to $\|\cdot\|$ on $H_\rho^\alpha(\mathbb{R}^n)$. Now, arguing by contradiction we suppose that $\{u_k\}$ is unbounded. Then, up to a subsequence, we may assume that

$$\|u_k\| \rightarrow \infty \text{ as } k \rightarrow \infty.$$

Thus

$$c = \lim_{k \rightarrow \infty} \left(I(u_k) - \frac{1}{2} I'(u_k) u_k \right) = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \left(\frac{1}{2} f(u_k(x)) u_k(x) - F(u_k(x)) \right) dx. \quad (3.5)$$

Let $w_k = \frac{u_k}{\|u_k\|}$, then $\{w_k\}$ is bounded in $H_\rho^\alpha(\mathbb{R}^n)$. We claim that,

$$\lim_{k \rightarrow \infty} \sup_{y \in \mathbb{R}^n} \int_{B(y, 2)} |w_k(x)|^2 dx = 0. \quad (3.6)$$

Otherwise, for some $\delta > 0$, up to a subsequence we have

$$\sup_{y \in \mathbb{R}^n} \int_{B(y, 2)} |w_k(x)|^2 dx \geq \delta > 0.$$

Let $z_k \in \mathbb{R}^n$ such that

$$\int_{B(z_k, 2)} |w_k(x)|^2 dx \geq \tau := \frac{\delta}{2} > 0 \quad (3.7)$$

and $v_k(x) = w_k(x + z_k)$. By the change of variable $\tilde{x} = x + y_k$, we find

$$\|w_k\|_0 \leq \|v_k\| \leq \|w_k\|_*$$

from where it follows that $\{v_k\}$ is also bounded in $H_\rho^\alpha(\mathbb{R}^n)$. Passing to a subsequence, we obtain

$$v_k \rightarrow v \text{ in } L_{loc}^p(\mathbb{R}^n) \text{ and } v_k(x) \rightarrow v(x) \text{ a.e. } x \in \mathbb{R}^n.$$

Since

$$\int_{B(0,2)} |v_k(x)|^2 dx = \int_{B(z_k,2)} |w_k(x)|^2 dx \geq \tau > 0, \quad (3.8)$$

we see that $v \neq 0$. Let $\tilde{u}_k(x) = \|u_k\|v_k(x)$. If $v(x) \neq 0$, we have the limit $|\tilde{u}_k(x)| \rightarrow +\infty$ which together with (f_3) leads to

$$\frac{F(\tilde{u}_k(x))}{|\tilde{u}_k(x)|^2} |v_k(x)|^2 \rightarrow +\infty. \quad (3.9)$$

The last limit combine with (3.9) to give

$$\begin{aligned} \frac{1}{2} - \frac{c + o(1)}{\|u_k\|^2} &= \int_{\mathbb{R}^n} \frac{F(u_k(x))}{\|u_k\|^2} dx \\ &= \int_{\mathbb{R}^n} \frac{F(\tilde{u}_k(x))}{\|u_k\|^2} dx \\ &\geq \int_{\{v \neq 0\}} \frac{F(\tilde{u}_k(x))}{|\tilde{u}_k(x)|} |v_k(x)|^2 dx \rightarrow +\infty, \end{aligned} \quad (3.10)$$

which is impossible. This shows (3.6). Then, by Lemma 2.1

$$w_k \rightarrow 0 \text{ in } L^q(\mathbb{R}^n), \quad \forall q \in (2, 2_\alpha^*). \quad (3.11)$$

We are going to get a contradiction as follow. By Property 3.1 - (1), given $\tau > 0$, there exists $C_\tau > 0$ such that

$$|F(t)| \leq \tau |t|^2 + C_\tau |t|^q. \quad (3.12)$$

Since $\|w_k\| = 1$, there exists a constant $K > 0$ such that

$$\|w_k\|_{L^2}^2 \leq K.$$

Therefore, by (3.11) and (3.12)

$$\limsup_{k \rightarrow \infty} \int_{\mathbb{R}^n} |F(w_k(x))| dx \leq \limsup_{k \rightarrow \infty} (\tau \|w_k\|_{L^2}^2 + C_\tau \|w_k\|_{L^q}^q) \leq \epsilon K.$$

Since τ is arbitrary, we deduce

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} F(w_k(x)) dx = 0. \quad (3.13)$$

Now, we choose a sequence $\{t_k\} \in [0, 1]$ such that

$$I(t_k u_k) = \max_{t \in [0,1]} I(t u_k). \quad (3.14)$$

Given $\sigma > 0$, noting that $\frac{(4\sigma)^{1/2}}{\|u_k\|} \in (0, 1)$ for k large enough, (3.13) ensures that

$$\begin{aligned} I(t_k u_k) &\geq I((4\sigma)^{1/2} w_k) = \frac{1}{2} \|(4\sigma)^{1/2} w_k\|^2 - \int_{\mathbb{R}^n} F((4\sigma)^{1/2} w_k(x)) dx \\ &= 2\sigma - \int_{\mathbb{R}^n} F((4\sigma)^{1/2} w_k(x)) dx \geq \sigma. \end{aligned}$$

Namely, $I(t_k u_k) \rightarrow +\infty$. But $I(0) = 0$ and $I(u_k) \rightarrow c$, then by (3.14) we see that $t_k \in (0, 1)$ and

$$\begin{aligned} 0 &= t_k \frac{d}{dt} I(tu_k) \Big|_{t=t_k} \\ &= \int_{\mathbb{R}^n} \int_{B(0, \rho(x))} \frac{|t_k u_k(x+z) - t_k u_k(x)|^2}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} V(x) |t_k u_k(x)|^2 dx \\ &\quad - \int_{\mathbb{R}^n} f(t_k u_k(x)) t_k u_k(x) dx. \end{aligned} \tag{3.15}$$

Now from (3.15) and (f_3) ,

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\frac{1}{2} f(u_k) u_k - F(u_k) \right) dx &\geq \frac{1}{\theta} \int_{\mathbb{R}^n} \left(\frac{1}{2} f(t_k u_k) t_k u_k - F(t_k u_k) \right) dx \\ &= \frac{1}{\theta} \left(\frac{1}{2} \|t_k u_k\|^2 - \int_{\mathbb{R}^n} F(t_k u_k) dx \right) \\ &= \frac{1}{\theta} I(t_k u_k) \rightarrow +\infty. \end{aligned}$$

This contradicts with (3.5). Thereby, $\{u_k\}$ is bounded. \square

4 Proof of Theorem 1.1

In the sequel, we will analysis the classes (ρ_1) , (ρ_2) and (ρ_3) separately.

4.1 Class 1: ρ is periodic

Let $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > 0$, then by the Ekeland variational principle, there is a sequence $\{u_k\}$ such that

$$I(u_k) \rightarrow c \text{ and } I'(u_k) \rightarrow 0.$$

By Lemma 3.2, $\{u_k\}$ is bounded in $H_\rho^\alpha(\mathbb{R}^n)$. In what follows, fix

$$\delta = \lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^n} \int_{B(y,2)} |u_k(x)|^2 dx. \tag{4.1}$$

If $\delta = 0$, the Lemma 2.1 yields

$$u_k \rightarrow 0 \text{ in } L^q(\mathbb{R}^n), \quad \forall q \in (2, 2_\alpha^*).$$

Then, arguing as in (3.13),

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} F(u_k(x)) dx = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f(u_k(x)) u_k(x) dx = 0. \quad (4.2)$$

The above limits together with (3.5) implies that $c = 0$, a contradiction. Therefore $\delta > 0$. So there exists a sequence $\{y_k\} \subset \mathbb{Z}^n$ and a real number $\tau > 0$ such that

$$\int_{B(0,2)} |v_k(x)|^2 dx = \int_{B(y_k,2)} |u_k(x)|^2 dx > \tau, \quad (4.3)$$

where $v_k(x) = u_k(x + y_k)$. Moreover, since $\|v_k\| = \|u_k\|$, going if necessary to a subsequence, there is $v \in H_\rho^\alpha(\mathbb{R}^n) \setminus \{0\}$ such that

$$v_k \rightharpoonup v \text{ in } H_\rho^\alpha(\mathbb{R}^n) \quad \text{and} \quad v_k \rightarrow v \text{ in } L_{loc}^p(\mathbb{R}^n);$$

Furthermore, by the \mathbb{Z}^n invariance of the problem, $\{v_k\}$ is also a $(PS)_c$ sequence of I . Thus for every $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$I'(v)\varphi = \lim_{k \rightarrow \infty} I'(u_k)\varphi = 0.$$

So $I'(v) = 0$ and v is a nontrivial weak solution of (P_ϵ) . Moreover, (f_4) together with Fatou's Lemma gives $I(v) \leq c$.

4.2 Class 2: ρ is asymptotically periodic

Hereafter, we denote by $I_\infty : H_{h_\infty}^\alpha(\mathbb{R}^n) \rightarrow \mathbb{R}$ the functional

$$I_\infty(u) = \frac{1}{2} \left(\int_{\mathbb{R}^n} \int_{B(0, h_\infty(x))} \frac{|u(x+z) - u(x)|^2}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} |u(x)|^2 dx \right) - \int_{\mathbb{R}^n} F(u(x)) dx$$

and by $w_\infty \in H_{h_\infty}^\alpha(\mathbb{R}^n)$ be a nontrivial critical point of I_∞ , which was obtained in the last subsection. Then,

$$I_\infty(w_\infty) \leq c_\infty \quad \text{and} \quad I'_\infty(w_\infty) = 0,$$

where c_∞ denotes the mountain pass level of I_∞ . Since we are assuming (f_4) , we know that

$$c_\infty = \inf_{u \in \mathcal{N}_\infty} I_\infty(u)$$

where

$$\mathcal{N}_\infty = \{u \in H_{h_\infty}^\alpha(\mathbb{R}^n) \setminus \{0\} : I'_\infty(u)u = 0\},$$

and so, $I_\infty(w_\infty) = c_\infty$. If c denotes the mountain pass level associated with I , the condition (ρ_3) gives $c \leq c_\infty$. Next, we will study the following situations:

$$c = c_\infty \quad \text{and} \quad c < c_\infty.$$

Case 1: $c = c_\infty$. As $\rho \leq h_\infty$ and $I'_\infty(w_\infty)w_\infty = 0$, we have that

$$I'(w_\infty)w_\infty \leq 0,$$

hence there is $t \in (0, 1]$ such that

$$tw_\infty \in \mathcal{N} = \{u \in H_\rho^\alpha(\mathbb{R}^n) \setminus \{0\} : I'(u)u = 0\}.$$

By (f_4) ,

$$c = \inf_{u \in \mathcal{N}} I(u),$$

then, as $t \in (0, 1]$,

$$c \leq I(tw_\infty) = I(tw_\infty) - \frac{1}{2}I'(tw_\infty)(tw_\infty) \leq I_\infty(w_\infty) - \frac{1}{2}I'_\infty(w_\infty)(w_\infty),$$

that is,

$$c \leq I_\infty(w_\infty) - \frac{1}{2}I'_\infty(w_\infty)(w_\infty) = I_\infty(w_\infty) = c_\infty.$$

Since we are supposing that $c = c_\infty$, we deduce that $u^* = tw_\infty$ verifies

$$I(u^*) = c \quad \text{and} \quad I'(u^*) = 0.$$

By (f_4) , it is easy to prove that u^* is a critical for I , which finishes the proof.

Case 2: $c < c_\infty$. Hereafter, we denote by $\{u_n\} \subset H_\rho^\alpha(\mathbb{R}^n)$ a sequence which satisfies

$$I(u_k) \rightarrow c \quad \text{and} \quad I'(u_k) \rightarrow 0.$$

By using standard arguments, we know that $\{u_k\}$ is a bounded sequence in $H_\rho^\alpha(\mathbb{R}^n)$. Hence, for some subsequence, there is $u \in H_\rho^\alpha(\mathbb{R}^n)$ such that

$$u_k \rightharpoonup u \quad \text{in} \quad H_\rho^\alpha(\mathbb{R}^n).$$

Claim: $u \neq 0$.

If $u = 0$, there are $R, \eta > 0$ and $\{y_k\} \subset \mathbb{R}^n$ such that

$$\limsup_{k \rightarrow +\infty} \int_{B_R(y_k)} |u_k|^2 dx \geq \eta. \quad (4.4)$$

Indeed, otherwise we must have

$$\lim_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_k|^2 dx = 0.$$

Then, by Lemma 2.1,

$$u_k \rightarrow 0 \quad \text{in} \quad L^q(\mathbb{R}^n) \quad \forall q \in (2, 2^*),$$

from where it follows that

$$\int_{\mathbb{R}^n} f(u_k) u_k dx \rightarrow 0.$$

The above limit together with $I'(u_k)u_k = o_n(1)$ implies that $u_k \rightarrow 0$ in $H_\rho^\alpha(\mathbb{R}^n)$, which contradicts the limit $I(u_k) \rightarrow c > 0$.

Setting $v_k(x) = u_n(x + y_k)$ and considering $y_k \in \mathbb{Z}^n$, we have that $\{v_k\}$ is bounded in $H_{h_\infty}^\alpha(\mathbb{R}^n)$ and there is $v \in H_{h_\infty}^\alpha(\mathbb{R}^n)$ such that

$$v_k \rightharpoonup v \quad \text{in} \quad H_{h_\infty}^\alpha(\mathbb{R}^n)$$

and

$$\int_{B_R(0)} |v|^2 dx \geq \eta > 0$$

which shows $v \neq 0$.

From (4.4), it is easy to see that $|y_k| \rightarrow +\infty$. Then, by (ρ_4)

$$\rho(x + y_k) \rightarrow h_\infty(x) \quad \forall x \in \mathbb{R}^n \quad \text{as} \quad k \rightarrow +\infty.$$

The above limit and $I'(u_k)(v(\cdot - y_k)) = o_k(1)$ combine to give

$$I'_\infty(v)v \leq 0.$$

Thus, there is $s \in (0, 1]$ such that $su \in \mathcal{N}_\infty$. Consequently,

$$c_\infty \leq I_\infty(sv) = I_\infty(sv) - \frac{1}{2}I'_\infty(sv)(sv) \leq I_\infty(v) - \frac{1}{2}I'_\infty(v)(v).$$

Since

$$I_\infty(v) - \frac{1}{2}I'_\infty(v)(v) = I(v) - \frac{1}{2}I'(v)(v)$$

it follows

$$c_\infty \leq I(v) - \frac{1}{2}I'(v)(v).$$

On the other hand, the Fatou's Lemma leads to

$$I(v) - \frac{1}{2}I'(v)(v) \leq \liminf_{k \rightarrow +\infty} (I(v_k) - \frac{1}{2}I'(v_k)(v_k)) = \liminf_{k \rightarrow +\infty} (I(u_k) - \frac{1}{2}I'(u_k)(u_k))$$

that is,

$$c_\infty \leq \liminf_{k \rightarrow +\infty} I(u_k) = c$$

which is a contradiction, because we are supposing $c < c_\infty$.

From this $u \neq 0$ and $I'(u) = 0$, which implies that I has a nontrivial critical point. Moreover, by Fatou's Lemma, it is possible to prove that $I(u) = c$.

4.3 Class 3: ρ has finite global minimum points

Hereafter, we will consider the following energy functional $J_\epsilon : H_\rho^\alpha(\mathbb{R}^n) \rightarrow \mathbb{R}$ defined by

$$J_\epsilon(u) = \frac{1}{2} \left(\int_{\mathbb{R}^n} \int_{B(0, \rho_\epsilon(x))} \frac{|u(x+z) - u(x)|^2}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} |u(x)|^2 dx \right) - \int_{\mathbb{R}^n} F(u(x)) dx.$$

It is easy to see that $J_\epsilon \in C^1(H_\rho^\alpha(\mathbb{R}^n), \mathbb{R})$ with

$$\begin{aligned} J'_\epsilon(u)v &= \int_{\mathbb{R}^n} \int_{B(0, \rho_\epsilon(x))} \frac{[u(x+z) - u(x)][v(x+z) - v(x)]}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} u(x)v(x) dx \\ &\quad - \int_{\mathbb{R}^n} f(u(x))v(x) dx, \end{aligned}$$

for any $u, v \in H_\rho^\alpha(\mathbb{R}^n)$. Thus, the critical points of J_ϵ are (weak) solutions of (P_ϵ) . Since the functional J_ϵ is not bounded from below on $H_\rho^\alpha(\mathbb{R}^n)$, we will work on *Nehari manifold* \mathcal{N}_ϵ associated with the functional J_ϵ , given by

$$\mathcal{N}_\epsilon = \{u \in H_\rho^\alpha(\mathbb{R}^n) \setminus \{0\} : J'_\epsilon(u)u = 0\}$$

and with the level

$$c_\epsilon = \inf_{u \in \mathcal{N}_\epsilon} J_\epsilon(u).$$

It is possible to prove that c_ϵ is the mountain pass level of functional J_ϵ , see Willem [30].

For $\rho \equiv 1$, we consider the problem

$$(-\Delta)_1^\alpha u + u = f(u) \text{ in } \mathbb{R}^n, \quad u \in H_1^\alpha(\mathbb{R}^n). \quad (P_\infty)$$

Associated with the problem (P_∞) , we have the energy functional $J_1 : H_\infty^\alpha(\mathbb{R}^n) \rightarrow \mathbb{R}$ given by

$$J_\infty(u) = \frac{1}{2} \left(\int_{\mathbb{R}^n} \int_{B(0,1)} \frac{|u(x+z) - u(x)|^2}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} |u(x)|^2 dx \right) - \int_{\mathbb{R}^n} F(u(x)) dx,$$

the level

$$c_\infty = \inf_{u \in \mathcal{M}_\infty} J_\infty(u)$$

and the Nehari manifold

$$\mathcal{M}_\infty = \{u \in H_\infty^\alpha(\mathbb{R}^n) \setminus \{0\} : J'_\infty(u)u = 0\}.$$

For $\rho \equiv \rho_\infty$, we fix the problem

$$(-\Delta)_{\rho_\infty}^\alpha u + u = f(u) \text{ in } \mathbb{R}^n, \quad u \in H_{\rho_\infty}^\alpha(\mathbb{R}^n), \quad (P_{\rho_\infty})$$

and as above, we denote by $J_{\rho_\infty}, c_{\rho_\infty}$ and $\mathcal{M}_{\rho_\infty}$ the energy functional, the mountain pass level and Nehari manifold associated with (P_{ρ_∞}) respectively.

The following result concerns the behavior of J_ϵ on \mathcal{M}_ϵ . Once its proof is standard, we omit it

Lemma 4.1. *The functional J_ϵ is bounded from below on \mathcal{M}_ϵ . Moreover, J_ϵ is coercive on \mathcal{N}_ϵ .*

As an immediate consequence of the last lemma, we have

Corollary 4.1. *Let $\{u_k\}$ be a sequence in \mathcal{N}_ϵ and $J_\epsilon(u_k) \rightarrow c_\epsilon$. Then $\{u_k\}$ is bounded in $H_\rho^\alpha(\mathbb{R}^n)$.*

The next theorem is a version of a result compactness on Nehari manifolds due to Alves [1] for regional fractional laplacian. It establishes that problem (P_∞) has a ground state solution.

Theorem 4.1. *Let $\{u_k\} \subset \mathcal{M}_\infty$ be a sequence with $J_\infty(u_k) \rightarrow c_\infty$. Then,*

I. $u_k \rightarrow u$ in $H_1^\alpha(\mathbb{R}^n)$,

or

II. There is $\{y_k\} \subset \mathbb{R}^n$ with $|y_k| \rightarrow +\infty$ and $w \in H_1^\alpha(\mathbb{R}^n)$ such that $w_k = u_k(\cdot + y_k) \rightarrow w$ in $H_1^\alpha(\mathbb{R}^n)$ and $J_\infty(w) = c_\infty$.

Proof. Similarly to Corollary 4.1, we can assume that $\{u_k\}$ is a bounded sequence, and so, there is $u \in H_1^\alpha(\mathbb{R}^n)$ and a subsequence of $\{u_k\}$, still denoted by itself, such that $u_k \rightharpoonup u$ in $H_1^\alpha(\mathbb{R}^n)$. Applying the Ekeland's variational principle, there is a sequence $\{w_k\}$ in \mathcal{M}_∞ with

$$w_k = u_k + o_k(1), \quad J_\infty(w_k) \rightarrow c_\infty$$

and

$$J'_\infty(w_k) - \tau_k E'_\infty(w_k) = o_k(1), \quad (4.5)$$

where $(\tau_k) \subset \mathbb{R}$ and $E_\infty(w) = J'_\infty(w)w$, for any $w \in H_1^\alpha(\mathbb{R}^n)$.

Since $\{w_k\} \subset \mathcal{M}_\infty$, (4.5) leads to

$$\tau_k E'_\infty(w_k)w_k = o_k(1).$$

Gathering (f_4) and Lemma 2.1, it is possible to prove that there is $\eta_1 > 0$ such that

$$E'_\infty(u)u \leq -\eta_1, \quad \forall u \in \mathcal{M}_\infty.$$

From this, $\tau_k \rightarrow 0$ as $k \rightarrow \infty$,

$$J_\infty(u_k) \rightarrow c_\infty \quad \text{and} \quad J'_\infty(u_k) \rightarrow 0.$$

Consequently, u is critical point of J_∞ .

Next, we will study the following possibilities: $u \neq 0$ or $u = 0$.

Case 1: $u \neq 0$.

By Fatou's Lemma , it is easy to check that

$$\begin{aligned}
c_\infty &\leq J_\infty(u) = J_\infty(u) - \frac{1}{\theta} J'_\infty(u)u \\
&= \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u\|^2 + \int_{\mathbb{R}^N} \left(\frac{1}{\theta} f(u)u - F(u)\right) dx \\
&\leq \liminf_{k \rightarrow \infty} \left\{ \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_k\|^2 + \int_{\mathbb{R}^N} \left(\frac{1}{\theta} f(u_k)u_k - F(u_k)\right) dx \right\} \\
&\leq \limsup_{k \rightarrow \infty} \left\{ \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_k\|^2 + \int_{\mathbb{R}^N} \left(\frac{1}{\theta} f(u_k)u_k - F(u_k)\right) dx \right\} \\
&= \lim_{k \rightarrow \infty} \left\{ J_\infty(u_k) - \frac{1}{\theta} J'_\infty(u_k)u_k \right\} = c_\infty.
\end{aligned}$$

Hence,

$$\|u_k\|^2 \rightarrow \|u\|^2 \quad \text{in } \mathbb{R},$$

from where it follows that $u_k \rightarrow u$ in $H_1^\alpha(\mathbb{R}^n)$.

Case 2: $u = 0$.

In this case, we claim that there are $R, \xi > 0$ and $\{y_k\} \subset \mathbb{R}^n$ satisfying

$$\limsup_{k \rightarrow \infty} \int_{B_R(y_k)} |u_k|^2 dx \geq \xi. \quad (4.6)$$

If the claim is false, we must have

$$\limsup_{k \rightarrow \infty} \sup_{y \in \mathbb{R}^n} \int_{B_R(y)} |u_k|^2 dx = 0.$$

Thus, by Lemma 2.1,

$$u_k \rightarrow 0 \text{ in } L^p(\mathbb{R}^n), \quad \forall p \in (2, 2_\alpha^*).$$

Recalling $J'_\infty(u_k)u_k = o_k(1)$, the last limit yields

$$\|u_k\|^2 \rightarrow 0,$$

or equivalently

$$u_k \rightarrow 0 \text{ in } H_1^\alpha(\mathbb{R}^n),$$

leading to $c_\infty = 0$, which is absurd. This way, (4.6) is true. Setting

$$w_k(x) = u_k(x + y_k),$$

we have that

$$J_\infty(w_k) = J_\infty(u_k) \quad \text{and} \quad \|J'_\infty(w_k)\| = \|J'_\infty(u_k)\|,$$

that is, $\{w_k\}$ is a sequence $(PS)_{c_\infty}$ for J_∞ . If $w \in H_1^\alpha(\mathbb{R}^n)$ denotes the weak limit of $\{w_n\}$, it follows from (4.6),

$$\int_{B_R(0)} |w|^2 dx \geq \xi,$$

and so, $w \neq 0$.

By repeating the same argument of the first case for the sequence $\{w_k\}$, we deduce that $w_k \rightarrow w$ in $H_1^\alpha(\mathbb{R}^n)$, $w \in \mathcal{M}_\infty$ and $J_\infty(w) = c_\infty$. \blacksquare

4.3.1 Estimates involving the minimax levels

The main goal of this section is to prove some estimates involving the minimax levels c_ϵ and c_∞ . First of all, we recall the inequality

$$J_\infty(u) \leq J_\epsilon(u) \quad \forall u \in H_\rho^\alpha(\mathbb{R}^n),$$

which implies

$$c_\infty \leq c_\epsilon, \quad \forall \epsilon > 0.$$

Lemma 4.2. *The minimax levels c_ϵ and c_{ρ_∞} satisfy the inequality $c_\epsilon < c_{\rho_\infty}$. Hence, $c_\infty < c_{\rho_\infty}$.*

Proof. In a manner analogous to Theorem 4.1, there is $U \in H_\rho^\alpha(\mathbb{R}^n)$ such that

$$J_{\rho_\infty}(U) = c_{\rho_\infty} \quad \text{and} \quad J'_{\rho_\infty}(U) = 0.$$

In the sequel, let $t > 0$ be satisfy $tU \in \mathcal{M}_\epsilon$. Thereby,

$$c_\epsilon \leq J_\epsilon(tU).$$

Since that by (ρ_5) , $\rho_\infty > \rho(x)$ for all $x \in \mathbb{R}^n$, we derive

$$c_\epsilon < J_{\rho_\infty}(tU) \leq \max_{s \geq 0} J_{\rho_\infty}(sU) = J_{\rho_\infty}(U) = c_{\rho_\infty}.$$

\blacksquare

Using the last lemma, we are able to prove that J_ϵ verifies the $(PS)_d$ condition for some values of d .

Lemma 4.3. *The functional J_ϵ satisfies the $(PS)_d$ condition for $d \leq c_\infty + \tau$, where $\tau = \frac{1}{2}(c_{\rho_\infty} - c_\infty) > 0$.*

Proof. Let $\{v_k\} \subset H_\rho^\alpha(\mathbb{R}^n)$ be a $(PS)_d$ sequence for functional J_ϵ with $d \leq c_\infty + \tau$. Similarly to Corollary 4.1, $\{v_k\}$ is a bounded sequence in $H_\rho^\alpha(\mathbb{R}^n)$, and so, for some subsequence, still denoted by $\{v_k\}$,

$$v_k \rightharpoonup v \text{ in } H_\rho^\alpha(\mathbb{R}^n),$$

for some $v \in H_\rho^\alpha(\mathbb{R}^n)$. Now, by using standard arguments, it is possible to prove that

$$J_\epsilon(v_k) - J_\epsilon(w_k) - J_\epsilon(v) = o_k(1) \quad (4.7)$$

and

$$\|J'_\epsilon(v_k) - J'_\epsilon(w_k) - J'_\epsilon(v)\| = o_k(1), \quad (4.8)$$

where $w_k = v_k - v$. Since $J'_\epsilon(v) = 0$ and $J_\epsilon(v) \geq 0$, from (4.7)-(4.8), $\{w_k\}$ is a $(PS)_{d^*}$ sequence for J_ϵ with $d^* = d - J_\epsilon(v) \leq c_\infty + \tau$.

Claim 1. *There is $R > 0$ such that*

$$\limsup_{k \rightarrow \infty} \sup_{y \in \mathbb{R}^n} \int_{B_R(y)} |w_k|^2 dx = 0.$$

If the claim is true, we have

$$\int_{\mathbb{R}^n} f(w_k) w_k dx \rightarrow 0.$$

On the other hand, by (4.8), we know that $J'_\epsilon(w_k) = o_k(1)$, then

$$\|w_k\|^2 \rightarrow 0$$

that is, $w_k \rightarrow 0$ in $H_\rho^\alpha(\mathbb{R}^n)$, and so, $v_k \rightarrow v$ in $H_\rho^\alpha(\mathbb{R}^n)$.

Proof of Claim 1: If the claim is not true, for each $R > 0$ given, we find $\xi > 0$ and $\{y_k\} \subset \mathbb{R}^n$ such that

$$\limsup_{k \rightarrow \infty} \int_{B_R(y_k)} |w_k|^2 \geq \xi > 0.$$

Using that $w_k \rightharpoonup 0$ in $H_\rho^\alpha(\mathbb{R}^n)$, it follows that $\{y_k\}$ is an unbounded sequence. Setting

$$\tilde{w}_k = w_k(\cdot + y_k),$$

we have that $\{\tilde{w}_k\}$ is bounded in $H_\rho^\alpha(\mathbb{R}^n)$. Thus, there are $\tilde{w} \in H_\rho^\alpha(\mathbb{R}^n) \setminus \{0\}$ and a subsequence of $\{\tilde{w}_k\}$, still denoted by itself, such that

$$\tilde{w}_k \rightharpoonup \tilde{w} \in H_\rho^\alpha(\mathbb{R}^n).$$

Moreover, since $J'_\epsilon(w_k)\phi(\cdot - y_k) = o_k(1)$ for each $\phi \in H_\rho^\alpha(\mathbb{R}^n)$, we obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \int_{B(0, \rho_\infty)} \frac{[\tilde{w}(x+z) - \tilde{w}(x)][\phi(x+z) - \phi(x)]}{|z|^{n+2\alpha}} + \int_{\mathbb{R}^n} \tilde{w}(x)\phi(x)dx \\ &\quad - \int_{\mathbb{R}^n} f(\tilde{w}(x))\phi(x)dx, \end{aligned}$$

from where it follows that \tilde{w} is a weak solution of (P_{ρ_∞}) . Consequently, after some routine calculations,

$$c_{\rho_\infty} \leq J_{\rho_\infty}(\tilde{w}) = J_{\rho_\infty}(\tilde{w}) - \frac{1}{\theta} J'_{\rho_\infty}(\tilde{w})\tilde{w} \leq \liminf_{k \rightarrow \infty} \left\{ J_\epsilon(w_n) - \frac{1}{\theta} J'_\epsilon(w_n)w_n \right\} = d^*,$$

that is, $c_{\rho_\infty} \leq c_\infty + \tau$, which is an absurd because $\tau < c_{\rho_\infty} - c_\infty$. Therefore, the Claim 1 is true. \square

In what follows, let us fix $\gamma_0, r_0 > 0$ such that

- $\overline{B_{\gamma_0}(a_i)} \cap \overline{B_{\gamma_0}(a_j)} = \emptyset$ for $i \neq j$ and $i, j \in \{1, \dots, \ell\}$
- $\bigcup_{i=1}^\ell B_{\gamma_0}(a_i) \subset B_{r_0}(0)$.
- $K_{\frac{\gamma_0}{2}} = \bigcup_{i=1}^\ell \overline{B_{\frac{\gamma_0}{2}}(a_i)}$

Besides this, we define the function $Q_\epsilon : H_\rho^\alpha(\mathbb{R}^n) \rightarrow \mathbb{R}^n$ by

$$Q_\epsilon(u) = \frac{\int_{\mathbb{R}^n} \chi(\epsilon x) |u|^2 dx}{\int_{\mathbb{R}^n} |u|^2 dx},$$

where $\chi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given by

$$\chi(x) = \begin{cases} x & \text{if } |x| \leq r_0 \\ r_0 \frac{x}{|x|} & \text{if } |x| > r_0. \end{cases}$$

The next two lemmas will be useful to get important (PS) -sequences associated with J_ϵ .

Lemma 4.4. *There are $\delta_0 > 0$ and $\epsilon_1 > 0$ such that if $u \in \mathcal{M}_\epsilon$ and $J_\epsilon(u) \leq c_\infty + \delta_0$, then*

$$Q_\epsilon(u) \in K_{\frac{\gamma_0}{2}} \text{ for } \epsilon \in (0, \epsilon_1).$$

Proof. If the lemma does not occur, there must be $\delta_k \rightarrow 0$, $\epsilon_k \rightarrow 0$ and $u_k \in \mathcal{M}_{\epsilon_k}$ such that

$$J_{\epsilon_k}(u_k) \leq c_\infty + \delta_k$$

and

$$Q_{\epsilon_k}(u_k) \notin K_{\frac{\gamma_0}{2}}.$$

Fixing $s_k > 0$ such that $s_k u_k \in \mathcal{M}_\infty$, we have

$$c_\infty \leq J_\infty(s_k u_k) \leq J_{\epsilon_k}(s_k u_k) \leq \max_{t \geq 0} J_{\epsilon_k}(t u_k) = J_{\epsilon_k}(u_k) \leq c_\infty + \delta_k.$$

Hence,

$$\{s_k u_k\} \subset \mathcal{M}_\infty \text{ and } J_\infty(s_k u_k) \rightarrow c_\infty.$$

Applying the Ekeland's variational principle, we can assume without loss of generality that $\{s_k u_k\} \subset \mathcal{M}_\infty$ is a sequence $(PS)_{c_\infty}$ for J_∞ , that is,

$$J_\infty(s_k u_k) \rightarrow c_\infty \text{ and } J'_\infty(s_k u_k) \rightarrow 0.$$

According to Theorem 4.1, we must consider the following cases:

i) $s_k u_k \rightarrow U \neq 0$ in $H_\rho^\alpha(\mathbb{R}^n)$;

or

ii) There exists $\{y_k\} \subset \mathbb{Z}^n$ with $|y_k| \rightarrow +\infty$ such that $v_k = s_k u_k(\cdot + y_k)$ is convergent in $H_\rho^\alpha(\mathbb{R}^n)$ for some $V \in H_\rho^\alpha(\mathbb{R}^n) \setminus \{0\}$.

By a direct computation, we can suppose that $s_k \rightarrow s_0$ for some $s_0 > 0$. Therefore, without loss of generality, we can assume that

$$u_k \rightarrow U \text{ or } v_k = u_k(\cdot + y_k) \rightarrow V \text{ in } H_\rho^\alpha(\mathbb{R}^n). \quad (4.9)$$

Analysis of i).

By Lebesgue's dominated convergence theorem

$$Q_{\epsilon_k}(u_k) = \frac{\int_{\mathbb{R}^n} \chi(\epsilon_k x) |u_k|^2 dx}{\int_{\mathbb{R}^n} |u_k|^2 dx} \rightarrow \frac{\int_{\mathbb{R}^n} \chi(0) |U|^2 dx}{\int_{\mathbb{R}^n} |U|^2 dx} = 0 \in K_{\frac{\gamma_0}{2}},$$

leading to $Q_{\epsilon_k}(u_k) \in K_{\frac{\gamma_0}{2}}$ for k large, which is absurd.

Analysis of ii).

From the equality $J'_{\epsilon_k}(u_k)(u_k) = 0$, we see that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \int_{B(0, \rho(\epsilon_k x + \epsilon_k y_k))} \frac{[v_k(x+z) - v_k(x)]^2}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} |v_k|^2(x) dx \\ &\quad - \int_{\mathbb{R}^n} f(\tilde{w}(x)) \phi(x) dx, \end{aligned}$$

Now, we will study two cases:

I) $|\epsilon_k y_k| \rightarrow +\infty$

and

II) $\epsilon_k y_k \rightarrow y$, for some $y \in \mathbb{R}^n$.

If I) holds, the limit (4.9) gives

$$\int_{\mathbb{R}^n} \int_{B(0, \rho_\infty)} \frac{[V(x+z) - V(x)]^2}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} |V|^2 dx - \int_{\mathbb{R}^n} f(V(x)) V(x) dx = 0,$$

and so, $V \in \mathcal{M}_\infty$. Thereby,

$$c_{\rho_\infty} \leq J_{\rho_\infty}(V) = J_{\rho_\infty}(V) - \frac{1}{\theta} J'_{\rho_\infty}(V) V \leq \liminf_{k \rightarrow \infty} \left\{ J_\infty(u_k) - \frac{1}{\theta} J'_\infty(u_k) u_k \right\} = c_\infty,$$

that is, $c_{\rho_\infty} \leq c_\infty$, which contradicts Lemma 4.2.

Now, if $\epsilon_k y_k \rightarrow y$ for some $y \in \mathbb{R}^n$, arguing as above we get

$$c_{\rho(y)} \leq c_\infty, \tag{4.10}$$

where $c_{\rho(y)}$ the mountain pass level of the functional $J_{\rho(y)} : H_\rho^\alpha(\mathbb{R}^n) \rightarrow \mathbb{R}$ given by

$$J_{\rho(y)}(u) = \frac{1}{2} \left(\int_{\mathbb{R}^n} \int_{B(0, \rho(y))} \frac{|u(x+z) - u(x)|^2}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} |u(x)|^2 dx \right) - \int_{\mathbb{R}^n} F(u(x)) dx.$$

Observe that

$$c_{\rho(y)} = \inf_{u \in \mathcal{M}_{\rho(y)}} J_{\rho(y)}(u)$$

where

$$\mathcal{M}_{\rho(y)} = \left\{ u \in H_\rho^\alpha(\mathbb{R}^N) \setminus \{0\} : J'_{\rho(y)}(u)u = 0 \right\}.$$

If $\rho(y) > 1$, a similar argument explored in the proof of Lemma 4.2 shows that $c_{\rho(y)} > c_\infty$, which contradicts the inequality (4.10). Therefore, $\rho(y) = 1$ and $y = a_i$ for some $i = 1, \dots, \ell$. Consequently,

$$\begin{aligned} Q_{\epsilon_k}(u_k) &= \frac{\int_{\mathbb{R}^n} \chi(\epsilon_k x) |u_k|^2 dx}{\int_{\mathbb{R}^n} |u_k|^2 dx} \\ &= \frac{\int_{\mathbb{R}^n} \chi(\epsilon_k x + \epsilon_k y_k) |v_k|^2 dx}{\int_{\mathbb{R}^n} |v_k|^2 dx} \rightarrow \frac{\int_{\mathbb{R}^n} \chi(y) |V|^2 dx}{\int_{\mathbb{R}^n} |V|^2 dx} = a_i \in K_{\frac{\gamma_0}{2}}. \end{aligned}$$

From this, $Q_{\epsilon_k}(u_k) \in K_{\frac{\gamma_0}{2}}$ for k large, which is a contradiction, since by assumption $Q_{\epsilon_k}(u_k) \notin K_{\frac{\gamma_0}{2}}$. \blacksquare

From now on, we will use the ensuing notation

- $\theta_\epsilon^i = \{u \in \mathcal{M}_\epsilon; |Q_\epsilon(u) - a_i| < \gamma_0\},$
- $\partial\theta_\epsilon^i = \{u \in \mathcal{M}_\epsilon; |Q_\epsilon(u) - a_i| = \gamma_0\},$
- $\beta_\epsilon^i = \inf_{u \in \theta_\epsilon^i} J_\epsilon(u)$

and

- $\tilde{\beta}_\epsilon^i = \inf_{u \in \partial\theta_\epsilon^i} J_\epsilon(u).$

The above numbers are very important in our approach, because we will prove that there is a (PS) sequence of J_ϵ associated with each θ_ϵ^i for $i = 1, 2, \dots, \ell$. To this end, we need of the following technical result

Lemma 4.5. *There is $\epsilon^* > 0$ such that*

$$\beta_\epsilon^i < c_\infty + \tau \quad \text{and} \quad \beta_\epsilon^i < \tilde{\beta}_\epsilon^i,$$

for all $\epsilon \in (0, \epsilon^*)$, where $\tau = \frac{1}{2}(c_{\rho_\infty} - c_\infty) > 0$.

Proof. From now on, $U \in H_\rho^\alpha(\mathbb{R}^n)$ is a ground state solution for J_∞ , that is,

$$J_\infty(U) = c_\infty \quad \text{and} \quad J'_\infty(U) = 0 \quad (\text{See Theorem 4.1}).$$

For $1 \leq i \leq \ell$, we define the function $\hat{U}_\epsilon^i : \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$\hat{U}_\epsilon^i(x) = U(x - \frac{a_i}{\epsilon}).$$

Claim 2. For all $i \in \{1, \dots, \ell\}$, we have that

$$\limsup_{k \rightarrow +\infty} (\sup_{t \geq 0} J_\epsilon(t\widehat{U}_\epsilon^i)) \leq c_\infty.$$

By change of variable gives

$$J_\epsilon(t\widehat{U}_\epsilon^i) = \frac{t^2}{2} \left(\int_{\mathbb{R}^n} \int_{B(0, \rho(\epsilon x + a_i))} \frac{|U(x+z) - U(x)|^2}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} |U(x)|^2 dx \right) - \int_{\mathbb{R}^n} F(tU(x)) dx.$$

Moreover, we know that there exists $s = s(\epsilon) > 0$ such that

$$\max_{t \geq 0} J_\epsilon(t\widehat{U}_\epsilon^i) = J_\epsilon(s\widehat{U}_\epsilon^i).$$

By a direct computation, it follows that $s(\epsilon) \not\rightarrow 0$ and $s(\epsilon) \not\rightarrow \infty$ as $\epsilon \rightarrow 0$. Thus, without loss of generality, we can assume $s(\epsilon) \rightarrow s_0 > 0$ as $\epsilon \rightarrow 0$. Thereby,

$$\limsup_{\epsilon \rightarrow 0} \left(\max_{t \geq 0} J_\epsilon(\widehat{U}_\epsilon^i) \right) \leq \frac{s_0^2}{2} \left(\int_{\mathbb{R}^n} \int_{B(0, \rho(\epsilon x + a_i))} \frac{|U(x+z) - U(x)|^2}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} |U(x)|^2 dx \right) - \int_{\mathbb{R}^n} F(s_0 U(x)) dx.$$

Consequently,

$$\limsup_{\epsilon \rightarrow 0} (\sup_{t \geq 0} J_\epsilon(t\widehat{U}_\epsilon^i)) \leq c_\infty \quad \text{for } i \in \{1, \dots, \ell\}.$$

Since $Q_\epsilon(\widehat{U}_\epsilon^i) \rightarrow a_i$ as $\epsilon \rightarrow 0$, then $\widehat{U}_\epsilon^i \in \theta_\epsilon^i$ for all ϵ small enough. On the other hand, by Claim 2, $J_\epsilon(\widehat{U}_\epsilon^i) < c_\infty + \frac{\delta_0}{4}$ holds also for ϵ small enough. This way, there exists $\epsilon^* > 0$ such that

$$\beta_\epsilon^i < c_\infty + \frac{\delta_0}{4}, \quad \forall \epsilon \in (0, \epsilon^*).$$

Thus, decreasing δ_0 if necessary, we can assume that

$$\beta_\epsilon^i < c_\infty + \tau, \quad \forall \epsilon \in (0, \epsilon^*).$$

In order to prove the other inequality, we observe that Lemma 4.4 yields $J_\epsilon(u) \geq c_\infty + \delta_0$ for all $u \in \partial\theta_\epsilon^i$ and $\epsilon \in (0, \epsilon^*)$. Therefore,

$$\tilde{\beta}_\epsilon^i \geq c_\infty + \frac{\delta_0}{2}, \text{ for } \forall \epsilon \in (0, \epsilon^*).$$

Thereby,

$$\beta_\epsilon^i < \tilde{\beta}_\epsilon^i,$$

for $\epsilon \in (0, \epsilon^*)$. ■

Lemma 4.6. *For each $1 \leq i \leq \ell$, there exists a $(PS)_{\beta_\epsilon^i}$ sequence, $\{u_k^i\} \subset \theta_\epsilon^i$ for functional J_ϵ .*

Proof. By Lemma 4.5, we know that $\beta_\epsilon^i < \tilde{\beta}_\epsilon^i$. Then, the lemma follows adapting the same ideas explored in [23]. ■

4.3.2 Conclusion of the proof for Class 3.

Let $\{u_k^i\} \subset \theta_\epsilon^i$ be a $(PS)_{\beta_\epsilon^i}$ sequence for functional J_ϵ given by Lemma 4.6. Since $\beta_\epsilon^i < c_\infty + \tau$, by Lemma 4.3 there is u^i such that $u_k^i \rightarrow u^i$ in $H_\rho^\alpha(\mathbb{R}^n)$. Thus,

$$u^i \in \theta_\epsilon^i, \quad J_\epsilon(u^i) = \beta_\epsilon^i \text{ and } J'_\epsilon(u^i) = 0.$$

Now, we infer that $u^i \neq u^j$ for $i \neq j$ as $1 \leq i, j \leq \ell$. To see why, it remains to observe that

$$Q_k(u^i) \in \overline{B_{\gamma_0}(a_i)} \text{ and } Q_k(u^j) \in \overline{B_{\gamma_0}(a_j)}.$$

Once

$$\overline{B_{\gamma_0}(a_i)} \cap \overline{B_{\gamma_0}(a_j)} = \emptyset \text{ for } i \neq j,$$

it follows that $u^i \neq u^j$ for $i \neq j$. From this, J_ϵ has at least ℓ nontrivial critical points for all $\epsilon \in (0, \epsilon^*)$, which proves the theorem. ■

5 A remark about the existence of Ground state solution

Now we are going to show that the problem (P_ϵ) has a ground state by supposing only $(f_1) - (f_3)$ and that ρ belongs to Class 1 or 2. Let

$$m = \inf_{\mathcal{O}} I(u), \tag{5.1}$$

where $\mathcal{O} = \{u \in H_\rho^\alpha(\mathbb{R}^n) \setminus \{0\} : I'(u) = 0\}$.

Suppose that u is an arbitrary critical point of I . By Property 3.1 - (2),

$$\mathcal{F}(t) \geq 0 \text{ for all } t \in \mathbb{R}. \quad (5.2)$$

Then

$$I(u) = I(u) - \frac{1}{2}I'(u)u = \frac{1}{2} \int_{\mathbb{R}^n} \mathcal{F}(u(x))dx \geq 0 \quad (5.3)$$

which implies that $m \geq 0$. Therefore $0 \leq m \leq I(v) < +\infty$. Let $\{u_k\} \subset \mathcal{O}$ be a sequence such that

$$I(u_k) \rightarrow m \text{ as } k \rightarrow \infty.$$

Then, for some $\beta > 0$ we have

$$\|u_k\| \geq \beta. \quad (5.4)$$

Arguing as in the proof of Lemma 3.2, $\{u_k\}$ is bounded in $H_\rho^\alpha(\mathbb{R}^n)$. Let δ as in (4.1) associated to $\{u_k\}$. If $\delta = 0$, then

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f(u_k(x))u_k(x)dx = 0,$$

and hence

$$\|u_k\|^2 = I'(u_k)u_k + \int_{\mathbb{R}^n} f(u_k(x))u_k(x)dx \rightarrow 0. \quad (5.5)$$

This contradicts with (5.4). Therefore $\delta > 0$ and there exists a sequence $\{y_k\} \subset \mathbb{Z}^n$ such that $v_k(x) = u_k(x + y_k)$ satisfies

$$I'(v_k) = 0 \quad \text{and} \quad I(v_k) = I(u_k) \rightarrow m \text{ as } k \rightarrow \infty,$$

and v_k converges weakly to some $v \neq 0$, a nonzero critical point of I . Furthermore, by (5.2) and Fatou's Lemma we deduce

$$\begin{aligned} I(v) &= I(v) - \frac{1}{2}I'(v)v = \frac{1}{2} \int_{\mathbb{R}^n} \mathcal{F}(v(x))dx \\ &\leq \liminf_{k \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^n} \mathcal{F}(v_k(x))dx \\ &= \liminf_{k \rightarrow \infty} \left(I(u_k) - \frac{1}{2}I'(u_k)u_k \right) = m. \end{aligned}$$

Therefore, v is a nontrivial critical point of I with $I(v) = m$.

Remark 5.1. We note that, by Theorem 1.1, v is a nontrivial solution but it is possible that $m = I(v) = 0$, because we are assuming that

$$\mathcal{F}(t) \geq 0 \quad \forall t \in \mathbb{R}.$$

To ensure that $m > 0$, it suffices to assume in addition that

$$\mathcal{F}(t) > 0, \quad \text{for } t \neq 0.$$

This is the case if f satisfies the following condition (f_4) . In fact

$$2F(t) = 2 \int_0^t \frac{f(s)}{s} s ds < 2 \int_0^t \frac{f(t)}{t} s ds = f(t)t, \quad (5.6)$$

which implies that $\mathcal{F}(t) > 0$. Furthermore, under this condition we can show that the mountain pass critical point is a ground state, namely

$$m = c = \inf_{u \in \mathcal{N}} I(u),$$

where

$$\mathcal{N} = \{u \in H_\rho^\alpha(\mathbb{R}^n) \setminus \{0\} : I'(u)u = 0\}.$$

In fact, by Remark 3.1, I has the mountain-pass geometry, and we can introduce the following class of paths:

$$\Gamma = \{\gamma \in C([0, 1], H_\rho^\alpha(\mathbb{R}^n)) : \gamma(0) = 0, \quad I(\gamma(1)) < 0\}.$$

The mountain-pass level

$$c = \inf_{\gamma \in \Gamma} \sup_{\sigma \in [0, 1]} I(\gamma(\sigma)) > 0$$

is therefore associated to Γ . Furthermore, by Remark 3.1 and following the ideas of [12], we can show that for any $u \in H_\rho^\alpha(\mathbb{R}^n) \setminus \{0\}$, there is a unique $t_u = t(u) > 0$ such that $t_u u \in \mathcal{N}$ and

$$I(t_u u) = \max_{t \geq 0} I(tu),$$

and we note that

$$m_* = \inf_{u \in H_\rho^\alpha(\mathbb{R}^n) \setminus \{0\}} \max_{t \geq 0} I(tu).$$

where

$$m_* = \inf_{u \in \mathcal{N}} I(u)$$

On the other hand, given any $u \in \mathcal{N}$, we may define the path $\gamma_u(t) = t(t_u u)$, where $T(t_u u) < 0$ and obtain that $\gamma_u \in \Gamma$. Thus, $c \leq m_*$.

The other inequality follows from the fact that, for any $\gamma \in \Gamma$, there exists $t \in (0, 1)$ such that $\gamma(t) \in \mathcal{N}$. To prove this fact, we note that if $I'(u)u \geq 0$, then, by (5.6) we get

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^n} F(u(x))dx \\ &= I'(u)u + \frac{1}{2} \int_{\mathbb{R}^n} \mathcal{F}(u(x))dx \\ &\geq 0 \end{aligned}$$

So, if we assume that $I'(\gamma(t))\gamma(t) > 0$ for all $t \in (0, 1]$, then $I(\gamma(t)) \geq 0$ for all $t \in (0, 1]$, contradicting $I(\gamma(1)) < 0$. In conclusion, we have proved that

$$m_* = c.$$

On the other hand,

$$m \geq m_* \quad \text{and} \quad c \geq m,$$

from where it follows that

$$m = m_* = c.$$

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